# Introduction to Angle

**Part - 01**

## Trigonometry

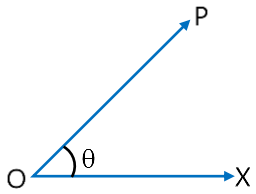
Introduction to Angle

Consider a revolving line OP.

Suppose that it revolves in anticlockwise direction starting from its initial position OX .

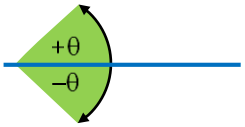
The angle is defined as the amount of revolution that the revolving line makes with its initial position.

From fig. the angle covered by the revolving line OP is $\theta = \angle POX$




The angle is taken positive if it is traced by the revolving line in anticlockwise direction.

The angle is taken negative if it is covered in clockwise direction.




1° = 60' (minute)

1' = 60" (second)

1 right angle = 90° (degrees) also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle by an arc of the circle, whose length is equal to the radius of the circle. 1 rad = $\frac{180^\circ}{\pi} \approx 57.3^\circ$

**Units of Angle**

Practical units : degrees (°)

1° = 60'(minute)

1' = 60"(second)

To convert an angle from degree to radian multiply it by $\frac{\pi}{180^\circ}$

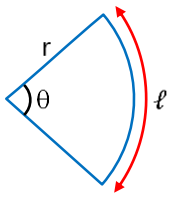
To convert an angle from radian to degree multiply it by $\frac{180^\circ}{\pi}$

**Relation between Angle and Arc**

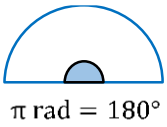
$$\text{Angle } (\theta) = \frac{\text{Arc}}{\text{Radius}} = \frac{\ell}{r}$$

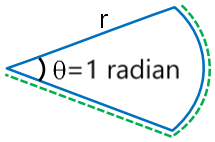
$\downarrow$

SI UNIT $\rightarrow$ Radian



**Radian**




$1\,rad = \frac{180^\circ}{\pi} \approx 57.3^\circ$

| | | |
|---|---|---|
| $2\pi = 360^\circ$ | $\pi = 180^\circ$ | $\frac{\pi}{2} = 90^\circ$ |
| $\frac{\pi}{3} = 60^\circ$ | $\frac{\pi}{4} = 45^\circ$ | $\frac{\pi}{6} = 30^\circ$ |
| $\frac{2\pi}{3} = 120^\circ$ | $\frac{3\pi}{4} = 135^\circ$ | $\frac{5\pi}{6} = 150^\circ$ |

**Illustration 1.**

Convert the given angles in desired units.

(i) 5° to minutes

(ii) 6' to seconds

(iii) 120" to minutes

**Solution.**

(i) 1° = 60'

5° × 60' = 300'

(ii) 1' = 60"

6' × 60" = 360"

(iii) 60" = 1'

$\frac{120''}{60''} = 2'$

**Illustration 2.**

Convert the given angles in desired units.

1. Convert 45° to radians

2. Convert $\frac{5\pi}{6}$ rad to degree

**Solution.**

1. $45^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{4}\text{radians}$

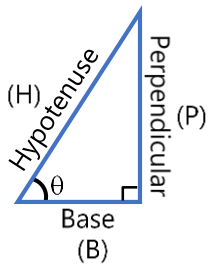
2. $\frac{5\pi}{6} \times \frac{180^\circ}{\pi} = 150^\circ$

# Pythagoras Theorem and Trigonometric Ratio

**Part - 02**

## Pythagoras Theorem

$P^2 + B^2 = H^2$




**Pythagorean Triplets**

| | |
|---|---|
| 3, 4, 5 | $(3^2 + 4^2 = 5^2)$ |
| 6, 8, 10 | $(6^2 + 8^2 = 10^2)$ |
| 7, 24, 25 | $(7^2 + 24^2 = 25^2)$ |
| 12, 16, 20 | $(12^2 + 16^2 = 20^2)$ |

Remember for fast calculations in Physics!!

**Trigonometric Ratios (or T ratios)**

$\sin\theta = \frac{P}{H}$ $\quad$ $\text{cosec}\theta = \frac{H}{P}$

$\cos\theta = \frac{B}{H}$ $\quad$ $\sec\theta = \frac{H}{B}$

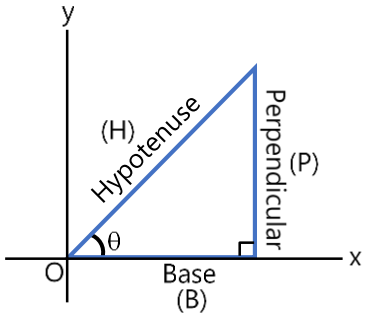
$\tan\theta = \frac{P}{B}$ $\quad$ $\cot\theta = \frac{B}{P}$

It can be easily proved that :

$\text{cosec}\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta}$



**Trigonometric Identities**

$\sin^2\theta + \cos^2\theta = 1$

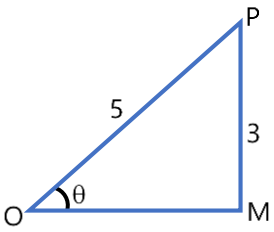
$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

**Illustration 1.**

Given $\sin\theta = 3/5$. Find all the other T-ratios, if $\theta$ lies in the first quadrant.

**Solution.**




In $\Delta OMP$, $\sin\theta = \frac{3}{5}$

so, MP = 3 and OP = 5

$\because$ $OM = \sqrt{(5)^2-(3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

Now, $\cos\theta = \frac{OM}{OP} = \frac{4}{5}$ $\tan\theta = \frac{MP}{OM} = \frac{3}{4}$

$\cot\theta = \frac{OM}{MP} = \frac{4}{3}$ $\sec\theta = \frac{OP}{OM} = \frac{5}{4}$ $\text{cosec}\theta = \frac{OP}{MP} = \frac{5}{3}$

**Table : The T-ratios of a few standard angles ranging from 0° to 90°**

| Angle($\theta$) | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ (not defined) |

# Quadrant Theory and Trigonometric Formulae

**Part - 03**

## Quadrants & ASTC Rule

In first quadrant, all trigonometric ratios are positive.
In second quadrant, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive.
In third quadrant, only $\tan\theta$ and $\cot\theta$ are positive.
In fourth quadrant, only $\cos\theta$ and $\sec\theta$ are positive

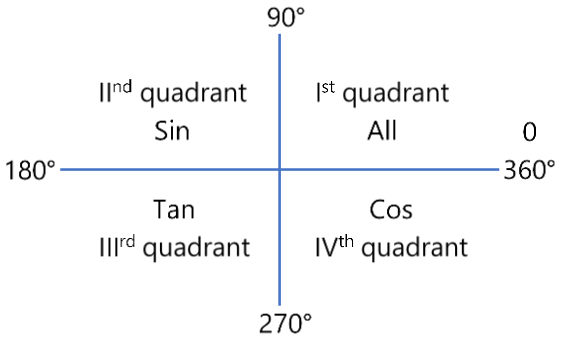
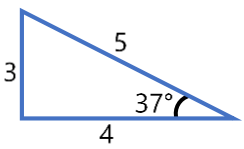



**Table to Remember**

| Angle($\theta$) | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ (not defined) |

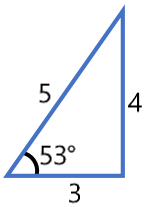
**T-Ratios of Special Angles**




$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$




$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

$\tan\theta = \frac{4}{3}$

T-ratios of angles greater than 90°

| | | |
|---|---|---|
| STEP-1 | Decide sign according to quadrant. | |
| STEP-2 | $\alpha = A \pm \theta$ | A : Integral multiple of 90°<br>$\theta$ : Acute angle |
| STEP-3 | If A is odd multiple of 90° | $\sin\theta \rightleftharpoons \cos\theta$<br>$\text{cosec}\theta \rightleftharpoons \sec\theta$<br>$\tan\theta \rightleftharpoons \cot\theta$ |
| | If A is Even multiple of 90° | No Change |

## Trigonometrical Ratios of General Angles (Reduction Formulae)

(i) Trigonometric function of an angle $(2n\pi + \theta)$ where n=0, 1, 2, 3,.... will be remain same.

$\sin(2n\pi + \theta) = \sin\theta$ $\quad \cos(2n\pi+\theta) = \cos\theta$ $\quad \tan(2n\pi+\theta)=\tan\theta$

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi-\theta) = +\sin\theta$ $\quad \cos(\pi-\theta) = -\cos\theta$ $\quad \tan(\pi-\theta)= -\tan\theta$

$\sin(\pi+\theta)= -\sin\theta$ $\quad \cos(\pi+\theta)= -\cos\theta$ $\quad \tan(\pi+\theta)= +\tan\theta$

$\sin(2\pi-\theta)= -\sin\theta$ $\quad \cos(2\pi-\theta)= +\cos\theta$ $\quad \tan(2\pi-\theta)= -\tan\theta$

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2}+\theta\right)=+\cos\theta$ $\quad \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$ $\quad \tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$

$\sin\left(\frac{\pi}{2}-\theta\right)=+\cos\theta$ $\quad \cos\left(\frac{\pi}{2}-\theta\right)=+\sin\theta$ $\quad \tan\left(\frac{\pi}{2}-\theta\right)=+\cot\theta$

(iv) Trigonometric function of an angle $-\theta$ (negative angles)

$\sin(-\theta) = -\sin\theta$ $\quad \cos(-\theta) = +\cos\theta$ $\quad \tan(-\theta) = -\tan\theta$

**Sum property for sine function**

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

**Sum property for cosine function**

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

**Sum property for tan function**

$$\tan(A + B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A - B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$$

**Double angle property**

$\sin2A = 2\sin A\cos A$

$\cos2A = \cos^2A - \sin^2A$ $\begin{cases} \cos2A = 2\cos^2A - 1 \\ \cos2A = 1 - 2\sin^2A \end{cases}$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

**Illustration 1.**

Write the function in terms of acute angle $\theta$

(1) $\cos(270° - \theta) = ?$

(2) $\cos(90° + \theta) = ?$

(3) $\cos\left(\frac{\pi}{2} - \theta\right) = ?$

(4) $\sin\left(\frac{\pi}{2} - \theta\right) = ?$

**Solution.**

(1) $\cos(270° - \theta) = \cos(\frac{3\pi}{2} - \theta) = -\sin\theta$

(2) $\cos(90° + \theta) = \cos(\frac{\pi}{2} + \theta) = -\sin\theta$

(3) $\cos(\frac{\pi}{2} - \theta) = +\sin\theta$

(4) $\sin(\frac{\pi}{2} - \theta) = +\cos\theta$

**Illustration 2.**

Evaluate :

(a) sin120° (b) tan150° (c) cos330°

**Solution.**

(a) $\sin120° = \sin(90° + 30°) = \sin(\frac{\pi}{2} + 30°) = +\cos30° = \frac{\sqrt{3}}{2}$

(b) $\tan150° = \tan(90° + 60°) = \tan(\frac{\pi}{2} + 60°) = -\cot60° = -\frac{1}{\sqrt{3}}$

(c) $\cos330° = \cos(360° - 30°) = \cos(2\pi - 30°) = +\cos30° = \frac{\sqrt{3}}{2}$

**Illustration 3.**

Evaluate :

(a) cos(–30°) (b) sin(–45°)

**Solution.**

(a) $\cos(-30°) = \cos(30°) = \frac{\sqrt{3}}{2}$

(b) $\sin(-45°) = -\sin(45°) = -\frac{1}{\sqrt{2}}$

**Illustration 4.**

Evaluate :

(a) sin105° (b) cos75°

**Solution.**

(a) $\sin(105°) = \sin(60° + 45°) = \sin60°\cos45° + \cos60°\sin45° = \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}$

(b) $\cos(75°) = \cos(45° + 30°) = \cos45°\cos30° - \sin45°\sin30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}$

**Illustration 5.**

Evaluate cos74°

**Solution.**

(a) $\cos74° = \cos(2 \times 37°)$

$= \cos^2(37°) - \sin^2(37°)$

$= \frac{16}{25}-\frac{9}{25}=\frac{7}{25}$

# Range of Trigonometric Functions

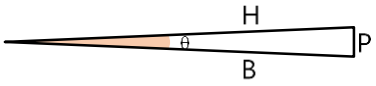
**Part - 04**

## Small angle approximation

$\sin\theta \approx \theta$

$\tan\theta \approx \theta$

$\cos\theta \approx 1$




Here $\theta$ must be in radians

### Illustration 1.

Find :

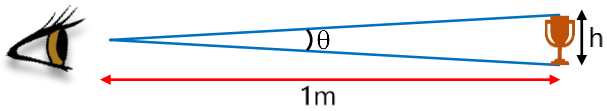
1. $\sin2°$ 2. $\tan1°$ 3. $\sin\pi°$

**Solution.**

1. $\sin2° = 2°\times\frac{\pi}{180°} = \frac{\pi}{90}$
2. $\tan1° = 1°\times\frac{\pi}{180°} = \frac{\pi}{180}$
3. $\sin\pi° = \pi\times\frac{\pi}{180}$

### Illustration 2.

A normal human eye can see an object making an angle 1.8° at the eye. What is the minimum height of object which can be seen by an eye from 1 m distance.




**Solution.**

$\theta$ is very small

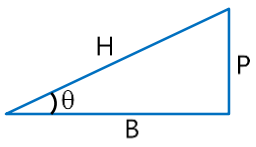
$\therefore \tan\theta \approx \theta$

$\theta = \frac{1.8°\times\pi}{180°} = \frac{\pi}{100}\text{rad}$

$\frac{h}{1} = \frac{\pi}{100}$

$\therefore$ h = 0.031 m

## Range of Trigonometric Functions



$\sin\theta = \frac{P}{H} \longrightarrow -1 \le \sin\theta \le 1$

$\cos\theta = \frac{B}{H} \longrightarrow -1 \le \cos\theta \le 1$

$\tan\theta = \frac{P}{B} \longrightarrow -\infty < \tan\theta < -\infty$

**Illustration 3.**

Find the maximum value of y = (sinx) (cosx)

(1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{\sqrt{2}}$ (4) $\sqrt{2}$

**Solution.**

y = sinxcosx = $\frac{1}{2}(\sin 2x) = \frac{1}{2}$ {sin2A = 2sinxcosx}

**Important result**

Range of function : "asinθ + bcosθ"

$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

**Illustration 4.**

If y = –3sinθ + 4 cosθ then find $y_{max}$ and $y_{min}$

**Solution.**

$-\sqrt{(-3)^2+(4)^2} \le y \le \sqrt{(-3)^2+(4)^2}$

$y_{max}$ = 5

$y_{min}$ = –5

# Co-ordinate Geometry

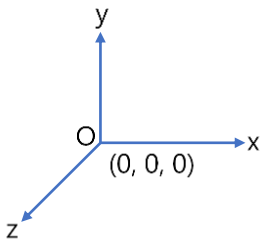
**Part - 05**

## Co-ordinate Geometry

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If a point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

- **Origin**
  This is any fixed point which is convenient to you. All measurements are taken w.r.t. this fixed point.

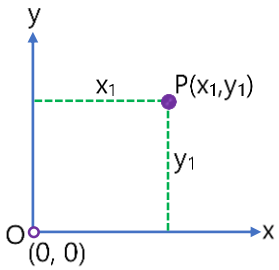
- **Axis or Axes**
  Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.



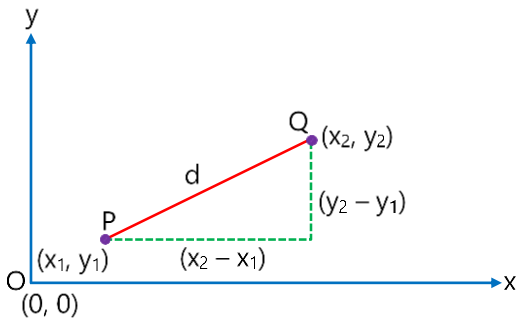

## Position of a point

$x_1$ = Abscissa : Distance of point from y axis

$y_1$ = Ordinate : Distance of point from x axis

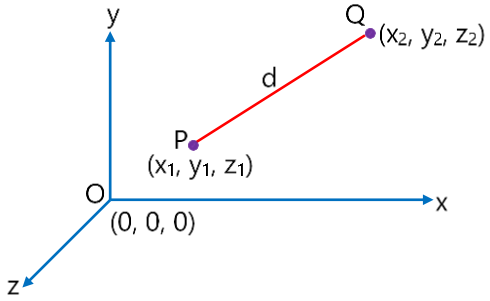



## Distance Formula in plane




In a plane : $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

## Distance Formula in space




In space : $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**Illustration 1.**

Find distance between two points A (1, 2, 5) and B (3, 4, 6).

**Solution.**

$A(x_1\ y_1\ z_1) = A(1, 2, 5)$

$B(x_2\ y_2\ z_2) = B(3, 4, 6)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(3-1)^2 + (4-2)^2 + (6-5)^2} = \sqrt{4+4+1} = 3$$

**Illustration 2.**

Find possible values of a if distance between the points (–9 cm, a cm) and (3 cm, 3 cm) is 13 cm.

**Solution.**

$$13 = \sqrt{(3+9)^2 + (3-a)^2}$$

$\Rightarrow$ $169 = 144 + (3 - a)^2$

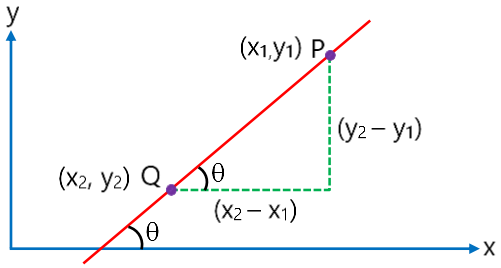
$\Rightarrow$ $\pm 5 = (3 - a)$

$\Rightarrow$ $+5 = 3 - a$ or $-5 = 3 - a$

$a = -2$cm or $a = 8$cm

## Slope of Line joining Two points

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$




Angle θ is measured from x axis.

### Illustration 4.

Find slope of a line passing through points A(2, 4) and B(3, 8)

### Solution.

$A(x_1, y_1) = A(2, 4)$

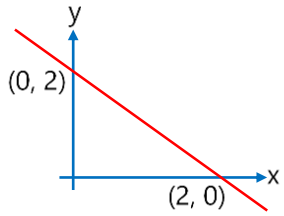
$B(x_2, y_2) = B(3, 8)$

$$\tan\theta = \frac{8-4}{3-2} = \frac{4}{1}$$

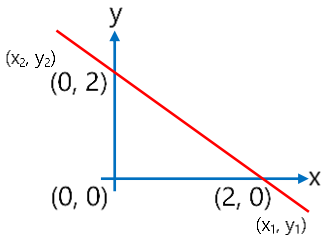
so, slope = 4

### Illustration 5.

Calculate slope of the shown line and its angle with x axis.




### Solution.




$$\tan\theta = \frac{2-0}{0-2}$$

$\tan\theta = -1$

So, slope = –1

angle with x-axis 135°.

# Equation of Straight Line

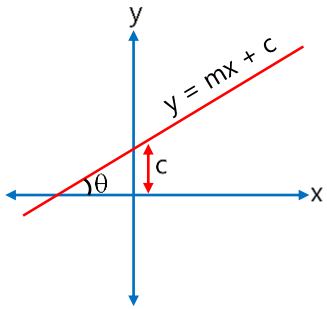
**Part - 06**

## Equation of Straight Line

Genera equation of straight line

$y = mx + c$

Slope or gradient = $\tan\theta$ Intercept on y-axis



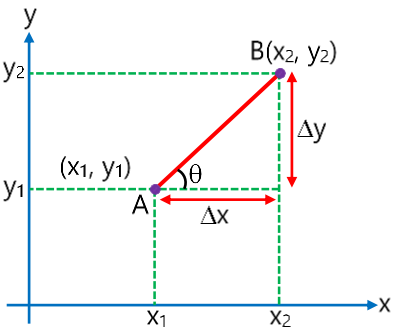

**Slope of A Line**

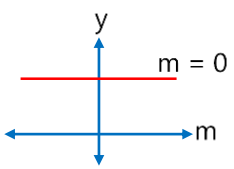
The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

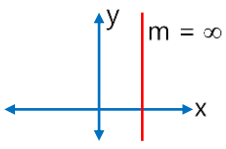
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$ [If both axes have identical scales]

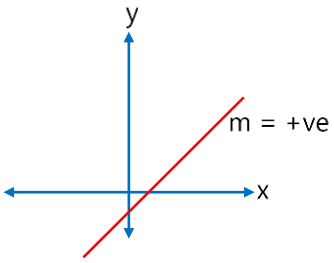
Here $\theta$ is the angle made by line with positive x-axis.

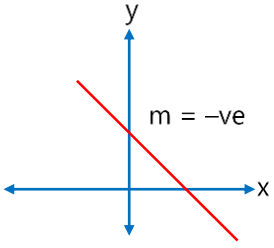
Slope of a line is a quantitative measure of inclination.



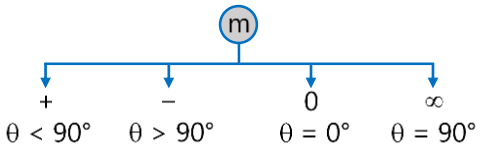



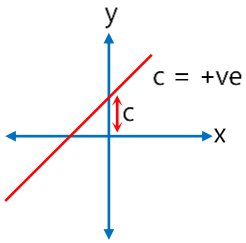



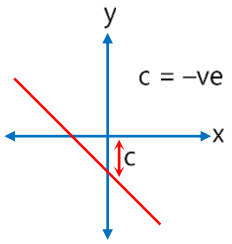



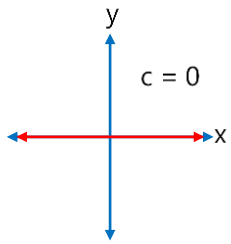
## Slope of a Straight line

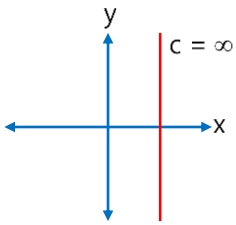



## Intercept of A Line



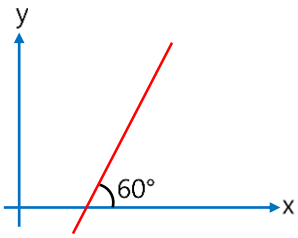




### Illustration 1.

Find slope of the line.




**Solution.**

slope = tanθ

= tan60°

= $\sqrt{3}$

### Illustration 2.

Find slope and intercept of a line y = 3x + 2, also draw the line.

**Solution.**

y = mx + c (general equation) ...(i)

y = 3x + 2 (given equation) ...(ii)

comparing equation (1) & (2)

m = slope = 3; c = 2

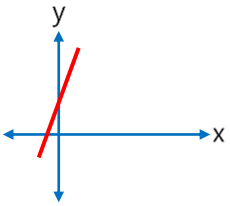


**Illustration 3.**

Find slope and intercept of a line 4y + 3x = 8, also draw the line.

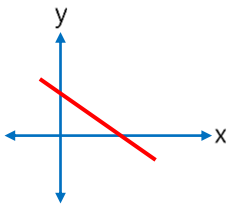
**Solution.**

y = mx + c (general equation) ...(i)

4y + 3x = 8 (given equation)

4y = – 3x + 8

$y = -\frac{3}{4}x+2$ ...(ii)

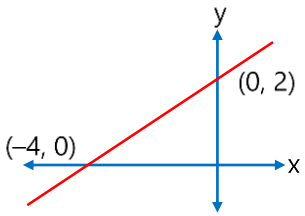



comparing equation (i) & (ii)

m = slope = $-\frac{3}{4}$; c = 2

**Illustration 4.**

Write equation of the line drawn




**Solution.**

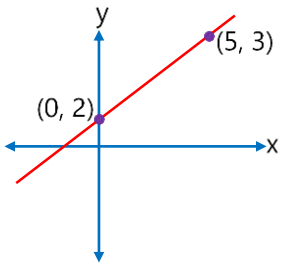
slope = $\tan\theta = \frac{2-0}{0+4} = \frac{1}{2}$

c = 2

$\therefore$ equation of straight line $y = \frac{1}{2}x+2$

**Illustration 5.**

Write equation of the line drawn




**Solution.**

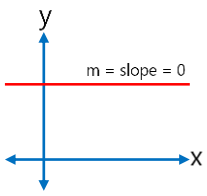
slope = $\tan\theta = \frac{3-2}{5-0} = \frac{1}{5}$

c = 2

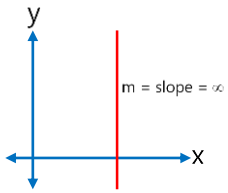
$\therefore$ equation of straight line $y = \frac{1}{5}x+2$

## Special Cases

1. Straight line parallel to x-axis




2. Straight line parallel to y-axis



# Introduction to Differentiation and Concept of Slope

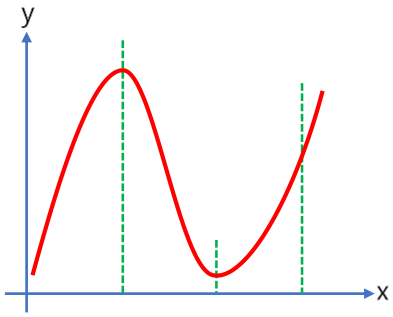
**Part - 07**

## Function

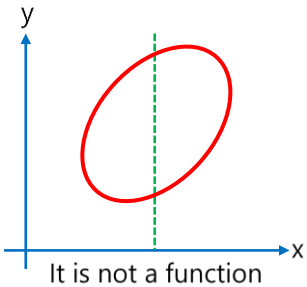
**Constant :** A quantity, whose value remains unchanged during mathematical operations, is called a constant quantity. The integers, fractions like $\pi$ ,e etc are all constants.

**Variable :** A quantity, which can take different values, is called a variable quantity. A variable is usually represented as x, y, z, etc.

**Function :** A quantity y is called a function of a variable x, if corresponding to any given value of x, there exists a single definite value of y. The phrase 'y is function of x' is represented as y = f (x)




(A) It is a function



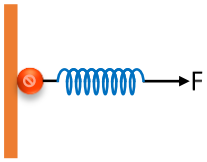

(B) It is not a function

## Finite change ($\Delta x$)

If change in a quantity is comparable to its initial value, change is said to be finite.

For example, if length of a spring is 10 cm, then a change of 2 cm in length is considered as finite change in length.

$L_o$ = 10 cm




$L_o$ = 10 cm

$\Delta L$ = 2 cm

## Infinitesimal change

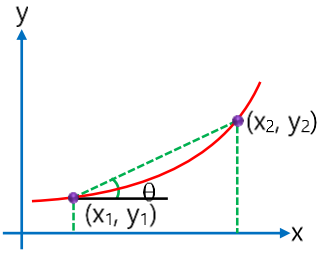
If change in a quantity is not comparable to its initial value, change is said to be infinitesimal.

- A very small change in y is called as dy
  for example $y_2$ = 100 and $y_1$ = 99.999999999.....
  $y_2 - y_1$ = dy = 0.0000000000.....001
- A very small change in x is called as dx
- A very small change in z is called as dz

➢ A very small change in t is called as dt

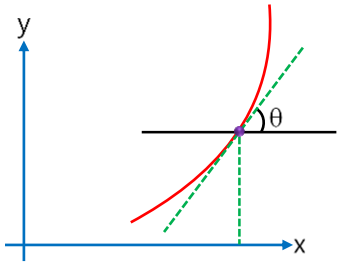
## Slope of A Curve

Average slope of curve
(Between two different points)



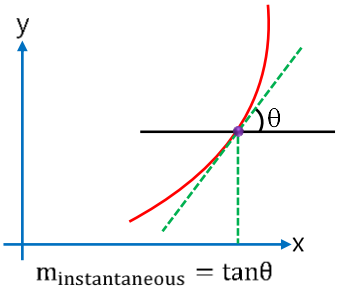

$$m_{avg} = \frac{\Delta y}{\Delta x} = \tan\theta$$

Instantaneous slope of curve
(At a Single point)




$$m_{instantaneous} = \tan\theta$$

**Instantaneous slope of curve**
(At a Single point)
To find the slope we require the tool called
DIFFERENTIATION




$$m_{instantaneous} = \tan\theta$$

## Definition of Differentiation/Derivative

At a point :

$\frac{dy}{dx}$ = "instantaneous rate of change of y w.r.t. x"

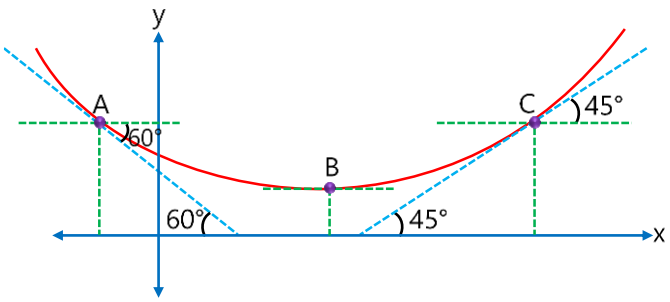
If y is a function of x : y = f(x)

Then derivative of y "w.r.t "x is given by :

$$y' = f'(x) = \frac{dy}{dx}$$

## Illustration 1.

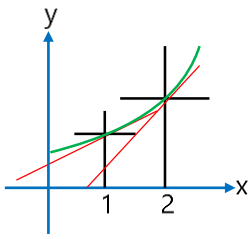
Find slope at A, B & C



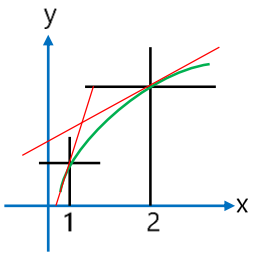

**Solution.**

| at A | at B | at C |
|---|---|---|
| slope = tanθ | slope = tanθ | slope = tanθ |
| = tan ($120^0$) | = tan(0°) | = tan(45°) |
| = tan(90° + 30°) | = 0 | = 1 |
| = –cot(30°) | | |
| = $-\sqrt{3}$ | | |

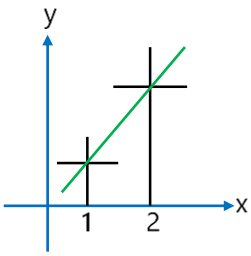
## Increasing functions




|Slope| → 1 < 2
|Slope| is increasing.

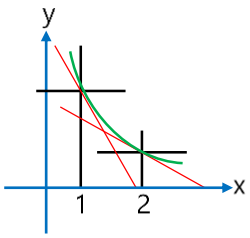



|Slope| → 1 > 2
|Slope| is decreasing.

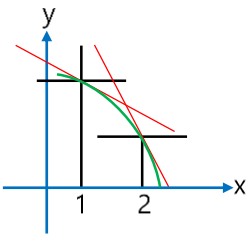



|Slope| → 1 = 2
|Slope| is constant.

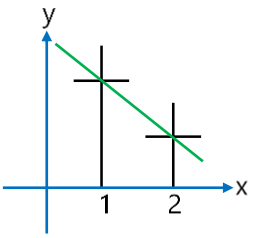
## Decreasing functions




|Slope| → 1 > 2
|Slope| is decreasing.



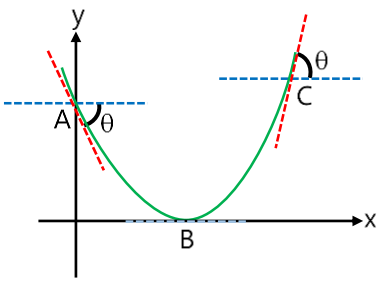

|Slope| → 1 < 2
|Slope| is increasing.




|Slope| → 1 = 2
|Slope| is constant.

**Illustration 2.**

Comment on slope and its magnitude from A to B & B to C



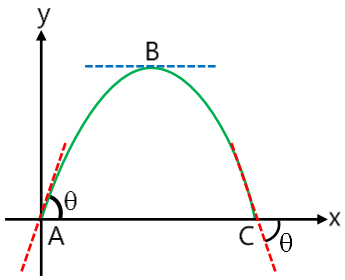

**Solution.**

A → B slope increasing  A → B magnitude decreasing

B → C slope increasing  B → C magnitude increasing

**Illustration 3.**

Comment on slope and its magnitude from A to B & B to C




**Solution.**

A → B slope decreasing  A → B magnitude decreasing

B → C slope decreasing  B → C magnitude increasing

# Differentiation of Standard Functions

**Part - 08**

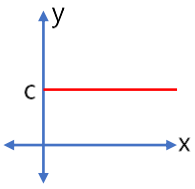
## Physical meaning of $\frac{dy}{dx}$

- The ratio of small change in the function y and the variable x is called the average rate of change of y w.r.t. x. For example, the velocity of a body changes by a small amount $\Delta v$ in small time $\Delta t$, then average acceleration of the body, $a_{av} = \frac{\Delta v}{\Delta t}$
- When $\Delta x \to 0$ The limiting value of $\frac{\Delta y}{\Delta x}$ is $\underset{\Delta t \to 0}{\text{Lim}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

## Derivative of Constant Functions

y = f (x) = constant, then it's derivative is ZERO.

$$\frac{dy}{dx} = \frac{dc}{dx} = 0$$




## Derivative of Algebraic Functions

y = f (x) = $x^n$, then it's derivative is :

$$\frac{dy}{dx} = \frac{dx^n}{dx} = nx^{(n-1)}$$

### Illustration 1.

Find the derivative of given functions w.r.t x

(1) $x^4$ (2) $x^{-2}$

**Solution.**

(1) $\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$

(2) $\frac{d}{dx}(x^{-2}) = -2x^{(-2-1)} = -2x^{-3}$

### Illustration 2.

Find value of $\frac{dy}{dx}$

(1) $y = x^6$ (2) $y = (x)^{\frac{3}{4}}$ (3) $y = \frac{1}{x}$

**Solution.**

(1) $\frac{d}{dx}(x^6)=6x^{6-1}=6x^5$

(2) $\frac{d}{dx}(x^{\frac{3}{4}})=\frac{3}{4}x^{\frac{3}{4}-1}=\frac{3}{4}x^{-\frac{1}{4}}$

(3) $\frac{d}{dx}(x^{-1})=-1x^{-1-1}=-\frac{1}{x^2}$

## Formulae for Differentiation of Trigonometric Functions

| | | |
|---|---|---|
| y = sinx | $\longrightarrow$ | $\frac{d(\sin x)}{dx}=\cos x$ |
| y = cosx | $\longrightarrow$ | $\frac{d(\cos x)}{dx}=-\sin x$ |
| y = tanx | $\longrightarrow$ | $\frac{d(\tan x)}{dx}=\sec^2 x$ |
| y = cosecx | $\longrightarrow$ | $\frac{d(\operatorname{cosec} x)}{dx}=-\operatorname{cosec} x\cot x$ |
| y = secx | $\longrightarrow$ | $\frac{d(\sec x)}{dx}=\sec x\tan x$ |
| y = cotx | $\longrightarrow$ | $\frac{d(\cot x)}{dx}=-\operatorname{cosec}^2 x$ |

## Formulae for Differentiation of Exponential Functions

| | | |
|---|---|---|
| $y = e^x$ | $\longrightarrow$ | $\frac{d(e^x)}{dx}=e^x$ |

## Formulae for Differentiation of Logarithmic Functions

| | | |
|---|---|---|
| $y = \log_e x$ or $\ln x$ | $\longrightarrow$ | $\frac{d(\ln x)}{dx}=\frac{1}{x}$ |

# Rules of Differentiation - Basic

**Part - 09**

## Constant Multiple Rule

If y = cf(x) = cU, then :

$$\frac{dy}{dx}=\frac{d(cU)}{dx}=c\frac{dU}{dx}$$

### Illustration 1.

Find the derivative of given functions w.r.t x

(1) $7x^6$ (2) $3x^{-2}$

**Solution.**

(1) $\frac{d}{dx}(7x^6)=7\frac{d}{dx}(x^6)=7\times6x^{6-1}=42x^5$

(2) $\frac{d}{dx}(3x^{-2})=3\frac{d}{dx}(x^{-2})=3\times(-2x^{-2-1})=-6x^{-3}$

### Illustration 2.

Find the derivative of given functions w.r.t x

(1) 3(sinx) (2) 4π(tanx)

**Solution.**

(1) $\frac{d}{dx}\left(3(\sin x)\right)=3\frac{d}{dx}(\sin x)=3\cos x$

(2) $\frac{d}{dx}\left(4\pi(\tan x)\right)=4\pi\frac{d}{dx}(\tan x)=4\pi\sec^2 x$

### Illustration 3.

Find the derivative of given functions w.r.t x

(1) 20(lnx) (2) $0.6(e^x)$

**Solution.**

(1) $\frac{d}{dx}\left(20(\ell n x)\right)=20\frac{d}{dx}(\ell nx)=\frac{20}{x}$

(2) $\frac{d}{dx}\left(0.6(e^x)\right)=0.6\frac{d}{dx}(e^x)=0.6e^x$

## Addition/Subtraction Rule

If U and V are functions of x and y is sum of functions U and V :

$$y = U \pm V$$

$$\frac{d(U\pm V)}{dx}=\frac{dU}{dx}\pm\frac{dV}{dx}$$

**Illustration 4.**

Find value of $\frac{dy}{dx}$

(1) $y = x^2 + 2x$ (2) $y = x^3 - x^{-2} + 1$

**Solution.**

(1) $\frac{d}{dx}(x^2+2x)=\frac{d}{dx}(x^2)+\frac{d}{dx}(2x)=2x+2$

(2) $\frac{d}{dx}(x^3-x^{-2}+1)=\frac{d}{dx}(x^3)-\frac{d}{dx}(x^{-2})+\frac{d}{dx}(1)=3x^2+2x^{-3}+0$

**Illustration 5.**

Find value of $\frac{dx}{dt}$ if : $x = 4t^2 + 3t + 2$

**Solution.**

$\frac{d}{dt}(4t^2+3t+2)=\frac{d}{dx}(4t^2)+\frac{d}{dx}(3t)+\frac{d}{dx}(2)=8t+3$

**Illustration 6.**

Find value of $\frac{dy}{dx}$ if : $y=\sqrt{x}+\frac{1}{\sqrt{x}}+1$

**Solution.**

$y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}+1$

$\frac{d}{dx}(x^{\frac{1}{2}}+x^{-\frac{1}{2}}+1)=\frac{d}{dx}(x^{\frac{1}{2}})+\frac{d}{dx}(x^{-\frac{1}{2}})+\frac{d}{dx}(1)$

$=\frac{1}{2}x^{\frac{1}{2}-1}-\frac{1}{2}x^{-\frac{1}{2}-1}+0$

$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$

**Illustration 7.**

Find value of $\frac{dy}{dx}$

(1) $y = 3\sin x + \cos x$ (2) $y = 2\tan x - x^3$

**Solution.**

$\frac{d}{dx}(3\sin x+\cos x)=\frac{d}{dx}(3\sin x)+\frac{d}{dx}(\cos x)=3\cos x-\sin x$

$\frac{d}{dx}(2\tan x-x^3)=\frac{d}{dx}(2\tan x)-\frac{d}{dx}x^3=2\sec^2 x-3x^2$

**Illustration 8.**

Find $\frac{dy}{dx}$ for the following

(i) $y=x^{\frac{7}{2}}$ (ii) $y = x^{-3}$ (iii) $y = x$ (iv) $y = x^5 + x^3 + 4x^{1/2} + 7$

(v) $y = 5x^4 + 6x^{3/2} + 9x$ (vi) $y = ax^2 + bx + c$ (vii) $y = 3x^5 - 3x - \frac{1}{x}$

**Solution.**

(1) $\frac{d}{dx}(x^{\frac{7}{2}}) = \frac{7}{2}x^{\frac{7}{2}-1} = \frac{7}{2}x^{\frac{5}{2}}$

(2) $\frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4}$

(3) $\frac{d}{dx}(x) = 1$

(4) $\frac{d}{dx}(x^5 + x^3 + 4x^{\frac{1}{2}} + 7) = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^3) + \frac{d}{dx}(4x^{\frac{1}{2}}) + \frac{d}{dx}(7)$

$= 5x^4 + 3x^2 + \frac{2}{\sqrt{x}} + 0$

(5) $\frac{d}{dx}(5x^4 + 6x^{\frac{3}{2}} + 9x) = \frac{d}{dx}(5x^4) + \frac{d}{dx}(6x^{\frac{3}{2}}) + \frac{d}{dx}(9x)$

$= 20x^3 + 9x^{\frac{1}{2}} + 9$

(6) $\frac{d}{dx}(ax^2 + bx + c) = \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx) + \frac{d}{dx}(c) = 2ax + b$

(7) $\frac{d}{dx}\left(3x^5 - 3x - \frac{1}{x}\right) = \frac{d}{dx}(3x^5) - \frac{d}{dx}(3x) - \frac{d}{dx}\left(\frac{1}{x}\right)$

$= 15x - 3 + \frac{1}{x^2}$

# Rules of Differentiation - Product Rule and Quotient Rule

**Part - 10**

## Product Rule

If we need to differentiate the product of two functions, then we apply product rule.

$$\frac{d(UV)}{dx} = V\frac{dU}{dx} + U\frac{dV}{dx}$$

### Illustration 1.

Find value of $\frac{dy}{dx}$

(1) $y = x \ln x$ (2) $y = e^x \cos x$

**Solution.**

(1) $\frac{d}{dx}(x\ell nx) = \ell nx\frac{d}{dx}(x) + x\frac{d}{dx}(\ell nx)$

$= \ell nx(1) + x\left(\frac{1}{x}\right) = \ell nx + 1$

(2) $\frac{d}{dx}(e^x \cos x) = \cos x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}\cos x$

$= \cos x e^x + e^x$

## Quotient Rule

If we need to differentiate a function which is the ratio of two functions, then we apply Quotient Rule.

$$\frac{d\left(\frac{U}{V}\right)}{dx} = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2}$$

### Illustration 2.

Find derivative of $y = \frac{4x}{x-7}$ w.r.t. x

**Solution.**

$$\frac{d\left(\frac{4x}{x-7}\right)}{dx} = \frac{(x-7)\frac{d}{dx}(4x) - 4x\frac{d}{dx}(x-7)}{(x-7)^2}$$

$$= \frac{(x-7)(4) - 4x(1)}{(x-7)^2} = -\frac{28}{(x-7)^2}$$

### Illustration 3.

Find value of $\frac{dy}{dx}$

(1) $y = x^2 \cos x$ (2) $y = e^x \sin x$

**Solution.**

(1) $\frac{d}{dx}(x^2\cos x)=\cos x\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(\cos x)=(2x)(\cos x)-x^2\sin x$

(2) $\frac{d}{dx}(e^x\sin x)=\sin x\frac{d}{dx}+e^x\frac{d}{dx}(\sin x)$

$= e^x\sin x+e^x\cos x$

**Illustration 4.**

Find value of $\frac{dy}{dx}$

(1) $y=\frac{x^2}{\cos x}$ (2) $y=\frac{\cot x}{x^3}$

**Solution.**

$$\frac{\cos x\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(\cos x)}{(\cos x)^2}=\frac{(\cos x)(2x)+(x^2)(\sin x)}{\cos^2 x}$$

**Illustration 5.**

Find value of $\frac{dI}{dt}$ if : $I = t^3(t - 2)$

**Solution.**

$I=t^4-2t^3$

$\frac{d}{dt}(t^4-2t^3)=4t^3-6t^2$

**Illustration 6.**

Find the derivative of q w.r.t. t if : $q = (t + 5)^3(t + 2)^4$

**Solution.**

$\frac{dq}{dt}=\frac{d}{dt}((t+5)^3(t+2)^4)$

$= (t + 2)^4(3)(t + 5)^2(1) + (t + 5)^3(4)(t + 2)^3(1)$

$= (t + 2)^3(t + 5)^2[3(t + 2) +4(t + 5)]$

# Rules of Differentiation - Chain Rule

**Part - 11**

## Chain Rule

The Chain Rule tells us how to find the derivative of a composite function.

If y = f(g(x)) then $\frac{dy}{dx}$ will be given by :

$$\frac{dy}{dx}=\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$$

Another way to represent the Chain Rule is :

If y = f(U) then $\frac{dy}{dx}$ will be given by :

$$\frac{dy}{dx}=\frac{df(U)}{dU}\times\frac{dU}{dx}$$

### Illustration 1.

Find value of $\frac{dy}{dx}$

(1) y = cos(2x + 3) (2) y = sin($x^2$ + $x^3$)

### Solution.

(1) Let u = 2x + 3 ∴ y = cos(u)

$$\frac{du}{dx}=\frac{d}{dx}(2x+3)=2 \qquad \frac{dy}{du}=\frac{d}{du}(\cos(u))=-\sin u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-(\sin u)(2)=-2\sin(2x+3)$$

(2) Let u = $x^2$ + $x^3$ ∴ y = sin(u)

$$\frac{du}{dx}=\frac{d}{dx}(x^2+x^3)=2x+3x^2 \qquad \frac{dy}{du}=\frac{d}{du}(\sin(u))=\cos u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\cos u(2x+3x^2)=\cos(x^2+x^3)(2x+3x^2)$$

### Illustration 2.

Find derivative of y w.r.t. x if : y = ln($x^3$ + 4)

### Solution.

Let u = $x^3$ + 4 ∴ y = ln(u)

$$\frac{du}{dx}=\frac{d}{dx}(x^3+4)=3x^2 \qquad \frac{dy}{du}=\frac{d}{du}(\ln(u))=\frac{1}{u}$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\left(\frac{1}{u}\right)(3x^2)=\frac{3x^2}{x^3+4}$$

**Illustration 3.**

Find value of $\frac{dy}{dx}$

$y = e^{(3x-6)}$

**Solution.**

Let $u = 3x - 6$ $\quad\therefore\quad$ $y = e^u$

$\frac{du}{dx} = \frac{d}{dx}(3x-6) = 3$ $\qquad$ $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \cdot 3 = 3e^{3x-6}$$

**Illustration 4.**

Find value of $\frac{dy}{dt}$

$y = A\sin(\omega t + \phi)$

**Solution.**

Let $u = \omega t + \phi$ $\quad\therefore\quad$ $y = A\sin u$

$\frac{du}{dx} = \frac{d}{dx}(\omega t + \phi) = \omega$ $\qquad$ $\frac{dy}{du} = \frac{d}{du}(A\sin u) = A\cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (A\cos u)(\omega) = A\omega\cos(\omega t + \phi)$$

**Illustration 5.**

Find value of $\frac{dy}{dt}$ if : $y = e^{\sin t}$

**Solution.**

Let $u = \sin t$ $\quad\therefore\quad$ $y = e^u$

$\frac{du}{dx} = \frac{d}{dx}(\sin t) = \cos t$ $\qquad$ $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (e^u)(\cos t) = e^{\sin t}\cos t$$

## Power Chain Rule

If $y = [f(x)]^n$ then $\frac{dy}{dx}$ will be given by :

$$\frac{dy}{dx} = n[f(x)]^{(n-1)} \frac{d[f(x)]}{dx}$$

$$\frac{dy}{dx} = n[f(x)]^{(n-1)} f'(x)$$

If $y = U^n$ then $\frac{dy}{dx}$ will be given by :

$$\frac{dy}{dx} = nU^{(n-1)} \frac{dU}{dx}$$

**Illustration 6.**

Find the derivative of $y = (x^2 + 3)^6$ w.r.t. x

**Solution.**

Let $u = x^2 + 3$ $\quad\therefore\quad$ $y = u^6$

$$\frac{du}{dx}=\frac{d}{dx}(x^2+3)=2x \qquad \frac{dy}{du}=\frac{d}{du}(u^6)=6u^5$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(6u^5)(2x)=6(x^2+3)^5(2x)$$

**Illustration 7.**

Find the derivative of : $x=\sqrt{t^2+2}$ w.r.t. t

**Solution.**

Let $u = t^2 + 2$ $\quad\therefore\quad$ $x=u^{\frac{1}{2}}$

$$\frac{du}{dt}=\frac{d}{dt}(t^2+2)=2t \qquad \frac{dx}{du}=\frac{d}{du}(u^{\frac{1}{2}})=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}$$

$$\frac{dx}{dt}=\frac{dx}{du}\times\frac{du}{dt}=\frac{1}{2\sqrt{u}}(2t)=\frac{t}{\sqrt{t^2+2}}$$

**Illustration 8.**

Find the derivative of y w.r.t. x

(1) $y = \cos^2 x$ (2) $y = \sin(x^2)$

**Solution.**

(1) $\cos 2x = 2\cos^2 x - 1$

$$\Rightarrow \frac{1+\cos 2x}{2}=\cos^2 x$$

$$\therefore\quad y=\frac{1+\cos 2x}{2}$$

$$=\frac{1}{2}+\frac{\cos 2x}{2}$$

$$\frac{dy}{dx}=0+\left(-\frac{\sin 2x}{2}\right)(2)=-\sin(2x)$$

(2) Let $u = x^2$ $\quad\therefore\quad$ $y = \sin(u)$

$$\frac{du}{dx}=\frac{d}{dx}(x^2)=2x \qquad \frac{dy}{du}=\frac{d}{du}(\sin u)=\cos u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\cos(u)(2x)=\cos(x^2)(2x)$$

# Applications of Derivatives

**Part - 12**

## Application of Derivatives

Instantaneous rate of change of a quantity "w.r.t." another quantity

$\vec{v}_{inst} = \frac{d\vec{x}}{dt}$

$\vec{a}_{inst} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}$

$\vec{F}_{inst} = \frac{d\vec{p}}{dt}$

### Illustration 1.

If height of magical tree depends on time as $h = 3t^2 + 5t + 2$ m. Find out :-

(1) Rate of change of height at t = 3 sec.

(2) Rate of change of height from t = 0 to t = 3 sec.

### Solution.

(1) $\frac{dh}{dt} = \frac{d}{dt}(3t^2 + 5t + 2) = 6t + 5$

$\frac{dh}{dt}$ = rate of change of height = 6t + 5

$\therefore$ rate of change of height at t = 3

= 6(3) + 5 = 23 m

(2) Height at t = 3, $h = 3(3^2) + 5(3) + 2 = 44$ m

height at t = 0, $h = 3(0^2) + 5(0) + 2 = 2$ m

$\frac{\Delta h}{\Delta t} = \frac{h_f - h_i}{t_f - t_i} = \frac{44-2}{3-0} = 14m$

### Illustration 2.

If position of particle is given by $x = (3t^2 + 4t - 1)$ m. Find its initial velocity and initial acceleration.

### Solution.

$\frac{dx}{dt} = \frac{t}{dt}(3t^2 + 4t - 1)$

= 6t + 4

$\Rightarrow$ Velocity = $\left.\frac{dx}{dt}\right_{(t=0)} = 6(0) + 4 = 4m/s$

$\frac{d^2x}{dt^2} = 6$

$\Rightarrow$ Acceleration = $6m/s^2$ (constant)

**Illustration 3.**

If position of particle is given by $x = (t^3 - 36t^2 + 30t - 1)$ m. Find its velocity when acceleration becomes zero.

**Solution.**

Velocity $= \frac{dx}{dt} = 3t^2 - 72t + 30$ m/s$^2$ ...(i)

acceleration $= \frac{d^2x}{dt^2} = 6t - 72$ m/s$^2$ ...(ii)

acceleration $= 6t - 72$ m/s$^2$ $= 0$

$\therefore t = 12$ s

Velocity at $t = 12$ s,

velocity $= 3(12)^2 - 72(12) + 30 = -402$ m/s

**Illustration 4.**

The area A of a circle is related to its radius by the equation $A = \pi r^2$. How fast is the area changing with respect to the radius when the radius is 10 m?

**Solution.**

$A = \pi r^2$

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi(2r)$$

$$= \pi(2)(10) = 20\pi \text{ m}$$

**Illustration 5.**

If side of a cube is changing by a rate of 4 m/s find rate of change of its volume w.r.t. time when side length is 2m.

**Solution.**

$V = a^3$

$$\frac{dV}{dt} = \frac{d}{dt}(a^3) = \frac{3a^2 da}{dt} = 3(2)^2(4) = 48 \text{ m}^3/\text{s}$$

**Illustration 6.**

The area of a block of ink is growing such that after t second its area is given by $A = (3t^2 + 7)$ cm$^2$. Calculate the rate of increase of area at = 5second.

**Solution.**

$$\frac{dA}{dt} = \frac{d}{dt}(3t^2 + 7) = 6t$$

at $t = 5$, $\frac{dA}{dt} = 6(5) = 30$ cm$^2$/s

## Application of Derivatives

### Slope of a curve at a given point

$$m_{inst} = \tan\theta = \frac{dy}{dx}$$

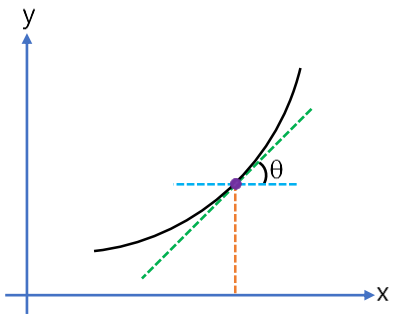



**Illustration 7.**

Find the slope of the tangent of a curve $y = x^2 + 2x + 4$ at $x = 0$ and $x = -1$

**Solution.**

$$\frac{dy}{dx}=\frac{d}{dx}(x^2+2x+4)=\frac{d}{dx}(x^2)+\frac{d}{dx}(2x)+\frac{d}{dx}(4)=2x+2$$

Slope of the tangent at $x = 0$, $\frac{dy}{dx} = 2(0) + 2 = 2$

Slope of the tangent at $x = -1$, $\frac{dy}{dx} = 2(-1) + 2 = 0$

# Concept of Maxima and Minima

**Part - 13**

## Double Differentiation

$$y = f(x) \xrightarrow{\text{Differentiation}} \frac{dy}{dx} \xrightarrow{\text{Differentiation}} \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$$

Function — First Derivative — Second Derivative

## Maxima and Minima

Suppose a quantity y depends on another quantity x in a manner shown in the figure.

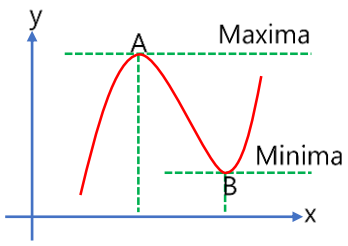
First derivative

$\frac{dy}{dx}$ = rate of change of y w.r.t. x = slope

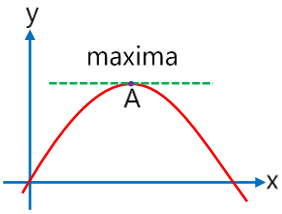
Second derivative

$\frac{d^2y}{dx^2}$ = rate of change of slope

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\text{slope})$$




## Maxima

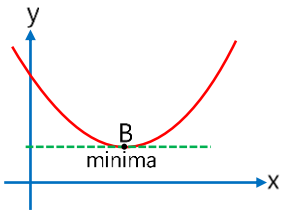



Condition for maxima are :

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

in figure at point A (maxima)

## Minima




Condition for minima are :

$\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$

in figure at point B (minima)

## Summary

**How to check Maxima & Minima of a function Y**

Step-1 : Calculate $\frac{dy}{dx}$ and put it equal to zero.

Step-2 : Find value of x from above equation.

Step-3 : Find $\frac{d^2y}{dx^2}$ to check for maxima and minima

- If $\frac{d^2y}{dx^2}>0$ "then minima"
- If $\frac{d^2y}{dx^2}<0$ then maxima
- If $\frac{d^2y}{dx^2}=0$ then neither maxima, nor minima

### Illustration 1.

Find maximum or minimum value for given equation $y = x^2 - 4x + 8$

**Solution.**

Step-1 : $\frac{dy}{dx}=2x-4$ ...(i)

Step-2 : $2x - 4 = 0$

$x = 2$

Step-3 : $\frac{d^2y}{dx^2}=2$ ...(ii)

$\frac{d^2y}{dx^2}>0$

So, minima at $x = 2$

$\therefore$ minimum value of given equation at $x = 2$; $y = (2)^2 - 4(2) + 8 = 4$

### Illustration 2.

Find local maximum and minimum value for $y = x^3 + 2x^2 - 4x + 2$

**Solution.**

Step-1 : $\frac{dy}{dx}=3x^2+4x-4$

Step-2 : $3x^2 + 4x - 4 = 0$

$\Rightarrow (3x - 2)(x + 2) = 0$

$x = \frac{2}{3}$ or $x = -2$

Step-3 : $\frac{d^2y}{dx^2} = 6x + 4$

at $x = \frac{2}{3}$; $\frac{d^2y}{dx^2} = 8 > 0$ (minima)

$y_{minimum} = \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2 = \frac{14}{27}$

at $x = -2$ ; $\frac{d^2y}{dx^2} = -8 < 0$ (maxima)

$y_{maximum} = (-2)^3 + 2(-2)^2 - 4(-2) + 2 = 10$

# Indefinite Integration

**Part - 14**

## Integration

In integral calculus, the differential coefficient of a function is given. We are required to find the function.

Integration is basically used for summation . $\sum$ is used for summation of discrete values, while $\int$ sign is used for continous function.

Reverse process of differentiation

If $F'(x)=f(x)$, then $\int f(x)dx=F(x)+c$

Here, c is called constant of integration or arbitrary constant

$$\int f(x)dx = F(x) + c$$

Integration symbol — Integrand — Variable of Integration — Constant of Integration

## Types of Integration

1. Indefinite Integration
2. Definite Integration

1. Indefinite Integration

$$\int f'(x)dx = f(x)+c$$

2. Definite Integration

$$\int_a^b f'(x)dx = [f(x)]_a^b = [f(b)-f(a)]$$

## Formulae of Integration for Algebraic function

$$\int x^n dx = \frac{x^{n+1}}{n+1}+c \qquad \text{(Provided } n \neq -1\text{)}$$

$$\int \frac{1}{x}dx = \ln x + c$$

$$\int k dx = kx + c$$

$$\int e^x dx = e^x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

**Illustration 1.**

Evaluate :

(i) $\int x^6 dx$ (ii) $\int x^3 dx$

**Solution.**

(i) $\int x^6 dx = \frac{x^{6+1}}{6+1} + c$

$= \left(\frac{x^7}{7}\right) + c$

(ii) $\int x^3 dx = \frac{x^{3+1}}{3+1}$

$= \frac{x^4}{4} + c$

**Illustration 2.**

Find $\int \frac{1}{\sqrt{x}} dx$

**Solution.**

$I = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx$

$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x} + c$

## Rules For Integration

**Rule 1** $\int (u \pm v) dx \Rightarrow \int u dx \pm \int v dx$

**Rule 2** $\int cf(x) dx \Rightarrow c \int f(x) dx$

**Illustration 3.**

$\int (4x^2 - 6x + 2) dx$

**Solution.**

$I = \int (4x^2 - 6x + +2) dx = \int 4x^2 dx - \int 6x dx + \int 2 dx$

$= 4\left[\frac{x^{2+1}}{2+1}\right] - 6\left[\frac{x^{1+1}}{1+1}\right] + 2\left[\frac{x^{0+1}}{0+1}\right] = \frac{4}{3}x^3 - 3x^2 + 2x + c$

**Illustration 4.**

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

**Solution.**

$$I=\int\left(x^{\frac{1}{2}}+\frac{1}{x^{\frac{1}{2}}}\right)^2 dx=\int\left[\left(x^{\frac{1}{2}}\right)^2+\left(\frac{1}{x^{\frac{1}{2}}}\right)^2+2\left(x^{\frac{1}{2}}\right)\left(\frac{1}{x^{\frac{1}{2}}}\right)\right]$$

$$=\int\left(x+\frac{1}{x}+2\right)dx=\int x dx+\int\frac{1}{x}dx+\int 2dx$$

$$=\frac{x^2}{2}+\ln x+2x+c$$

**Illustration 5.**

$$\int(\cos\theta-\sin\theta+3)d\theta$$

**Solution.**

$$I=\int(\cos\theta-\sin\theta+3)d\theta=\int\cos\theta d\theta-\int\sin\theta d\theta+\int 3d\theta$$

$$= \sin\theta + \cos\theta + 3\theta + c$$

**Illustration 6.**

$$\int(e^x+x^e+e^e)dx$$

**Solution.**

$$I=\int(e^x+x^e+e^e)dx=\int e^x dx+\int x^e dx+\int e^e dx$$

$$=e^x+\frac{x^{e+1}}{e+1}+e^e x + c$$

**Illustration 7.**

Integrate y w.r.t x, where $y=e^x-\frac{1}{x}+4$

**Solution.**

$$I=\int\left(e^x-\frac{1}{x}+4\right)dx=\int e^x dx-\int\frac{1}{x}dx+\int 4dx$$

$$=e^x - \ln x + 4x + c$$

## Linear Substitution in Integration Algebraic function

$$\int(ax+b)^n dx=\frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)}+c$$

$$\int\frac{1}{ax+b}dx=\frac{1}{a}\ln(ax+b)+c$$

$$\int \sin(ax+b)dx = -\frac{\cos(ax+b)}{a}+c$$

$$\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a}+c$$

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$$

**Illustration 8.**

(i) $\int \cos(2x+4)dx$ (ii) $\int \frac{1}{4t-2}dt$

**Solution.**

(i) $I = \int \cos(2x+4)dx = \frac{\sin(2x+4)}{2}+c$

(ii) $I = \int \frac{1}{4t-2}dt = \frac{\ln(4t-2)}{4}+c$

**Illustration 9.**

$\int e^{(-4x+3)}dx$

**Solution.**

$I = \int e^{(-4x+3)}dx = \frac{e^{(-4x+3)}}{-4}+c$

**Illustration 10.**

$\int (3x-4)^4 dx$

**Solution.**

$I = \int (3x-4)^4 dx = \frac{(3x-4)^{4+1}}{4+1}\cdot\frac{1}{3} = \frac{1}{15}(3x-4)^5$

**Illustration 11.**

$\int (\sin\omega t)dt$

**Solution.**

$I = \int (\sin\omega t)dt = -\frac{\cos(\omega t)}{\omega}+c$

# Definite Integration

**Part - 15**

## Definite Integration

When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

If $\frac{d}{dx}(f(x)) = f'(x)$, then

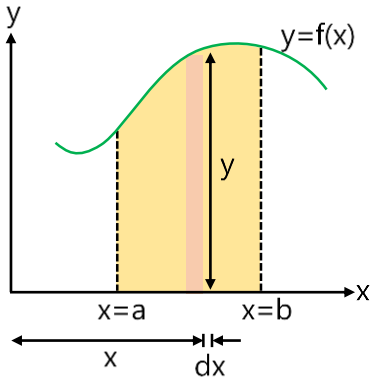
$\int f'(x)dx$ is called indefinite integral and $\int_a^b f'(x)dx$ is called definite integral

Here, a and b are called lower and upper limits of the variable x.

After carrying out integration, the result is evaluated between upper and lower limits as explained below :

$$\int_a^b f'(x)dx = |f(x)|_a^b = [f(b) - f(a)]$$

## Area under a curve and definite integration




Area of small shown darkly shaded element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then

$\int_a^b f(x)dx$ = shaded area between curve and x-axis.

$$\int_a^b f(x)dx = [F(x)]_a^b = [F(b) - F(a)]$$

### Illustration 1.

Find value of $\int_3^4 6xdx$

### Solution.

$$I=\int_3^4 6x\,dx=6\int_3^4 x\,dx=6\left[\frac{x^2}{2}\right]_3^4$$

$$=6\left[\frac{(4)^2}{2}-\frac{(3)^2}{2}\right]=6\left[\frac{7}{2}\right]=21$$

**Illustration 2.**

Find value of $\int_1^2(10x^2-4x+4)dx$

**Solution.**

$$I=\int_1^2(10x^2-4x+4)dx=\int_1^2 10x^2dx-\int_1^2 4x\,dx+\int_1^2 4\,dx$$

$$=10\left[\frac{(x^3)}{3}\right]_1^2-4\left[\frac{x^2}{2}\right]_1^2+4[x]_1^2$$

$$=10\left[\frac{2^3}{3}-\frac{1}{3}\right]-4\left[\frac{2^2}{2}-\frac{1}{2}\right]+4[2-1]=\frac{128}{6}$$

**Illustration 3.**

Find value of $\int_0^{\frac{\pi}{3}}\cos x\,dx$

**Solution.**

$$I=\int_0^{\frac{\pi}{3}}\cos x\,dx=[+\sin x]_0^{\frac{\pi}{3}}=\left(\sin\frac{\pi}{3}-\sin 0\right)$$

$$=\frac{\sqrt{3}}{2}$$

**Illustration 4.**

Find value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\,dx$

**Solution.**

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\,dx=[-\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\left[-\cos\left(\frac{\pi}{2}\right)-\left(-\cos\left(-\frac{\pi}{2}\right)\right)\right]=0$$

**Illustration 5.**

Find value of $\int_{\infty}^{0} e^{-2t}dt$

**Solution.**

$$I=\int_{\infty}^{0} e^{-2t}dt=\left[\frac{e^{-2t}}{-2}\right]_{\infty}^{0}=-\frac{1}{2}\left[e^{-2(0)}-e^{-2(\infty)}\right]=-\frac{1}{2}$$

**Illustration 6.**

Find value of $\int_{0}^{\frac{\pi}{6}} \sin 2\theta d\theta$

**Solution.**

$$I=\int_{0}^{\frac{\pi}{6}} \sin 2\theta d\theta=\left[-\frac{\cos 2\theta}{2}\right]_{0}^{\frac{\pi}{6}}$$

$$=-\frac{1}{2}\left[\cos\frac{2\pi}{6}-\cos 0\right]=\frac{1}{4}$$

# Applications of Integration - Analytical

**Part - 16**

## Application of Integration

There are many applications of integration such as :

(a) Displacement/Change in position $\Delta x = x_2 - x_1$

(b) Change in velocity $\Delta v = v_2 - v_1$

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt}$$

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt \qquad \int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt$$

(Change in position) $x_2 - x_1 = \int_{t_1}^{t_2} v\,dt$ (Change in velocity) $v_2 - v_1 = \int_{t_1}^{t_2} a\,dt$

### Illustration 1.

Initial position of a particle is x = 20 m and its velocity is $v = (2t^2 - 4t)$ m/s. Find position of the particle at t = 3 sec.

### Solution.

$$v = \frac{dx}{dt} \Rightarrow \int_{20}^{x} dx = \int_{t=0}^{t=3} v\,dt$$

$$\Rightarrow x - 20 = \int \left(2t^2 - 4t\right) dt$$

$$x - 20 = \left[\frac{2t^3}{3} - 2t^2\right]_0^3$$

$x - 20 = 18 - 18$

$x = 20$

### Illustration 2.

Initial velocity of a particle is '$2u_0$' and acceleration is a = kt. Find velocity at time t.

### Solution.

$$a = \frac{dv}{dt}$$

$$\int_{2u_0}^{v} dv = \int a\,dt = \int_{o}^{t} kt\,dt$$

$$[v]_{2u_0}^{v} = k\left[\frac{t^2}{2}\right]_0^t$$

$$v - 2u_0 = k\frac{t^2}{2}$$

$$v = \frac{kt^2}{2} + 2u_0$$

**Illustration 3.**

Find change in momentum from t = 1 to t = 2s if a force $F = 4t^2 - 6$ N acts on a particle. (Use $\Delta p = \int_{t_1}^{t_2} F dt$ )

**Solution.**

$$\Delta P = \int_{t_1}^{b_2} F dt$$

$$= \int_1^2 \left(4t^2 - 6\right) dt$$

$$= \int_1^2 \left(4t^2\right) dt - \int_1^2 6 dt$$

$$= \left[\frac{4t^3}{3}\right]_1^2 - [6t]_1^2$$

$$= \left[\frac{32}{3} - \frac{4}{3}\right] - [12 - 6]$$

$$= \frac{28}{3} - 6$$

$$= \frac{10}{3}$$

# Applications of Integration - Graphical

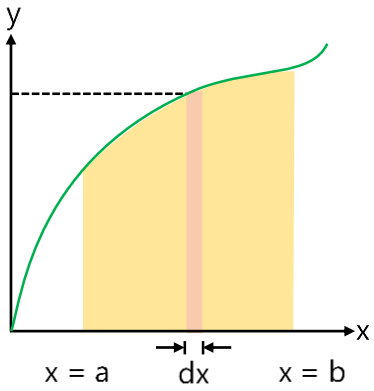
**Part - 17**

## Area Under The Curve

Area of shaded element small shown darkly = ydx = f(fx) dx

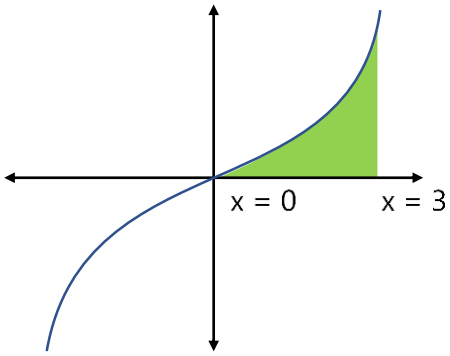
If we sum all areas between x = a and x = b then

$$\int_a^b f(x)dx = F(b) - F(a)$$




### Illustration 1.

Find area between the curve $y = x^3$ and x axis from x = 0 to x = 3.




### Solution.

Area under the curve = $\int ydx$

$$= \int_{x=0}^{x=3} x^3 dx$$

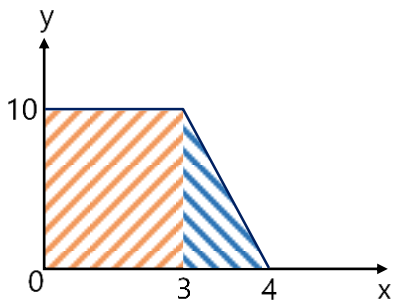
$$= \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{81}{4} \text{ unit}$$

**Illustration 2.**

Find the values of

(a) $\int_0^3 y\,dx$ (b) $\int_3^4 y\,dx$




**Solution.**

(a) $\int_0^3 y\,dx$ = Area under the curve from x = 0 to x = 3

= 10 × 3 = 30 unit

(b) $\int_3^4 y\,dx$ = Area under the wave from x = 3 to x = 4

$=\frac{1}{2}\times 1\times 10=5$ unit

**Illustration 3.**

Find area under the curve $y = 2x^2 - 4x + 6$, from x = 2 to x = 4.

**Solution.**

Area under the curve = $\int_{x_1}^{x_2} y\,dx=\int_2^4\left(2x^2-4x+6\right)dx$

$=\int_2^4 2x^2dx-\int_2^4 4xdx+\int_2^4 6dx$

$=\left[\frac{2x^3}{3}\right]_2^4-\left[2x^2\right]_2^4+\left[6x\right]_2^4$

$=\left(\frac{112}{3}\right)-(24)+12$

$=\frac{76}{3}$ unit

**Illustration 4.**

Find area under the curve y = cos x, from x = 0 to $x = \frac{\pi}{2}$.

**Solution.**

Area under of the curve = $\int y\,dx$

$=\int_0^{\frac{\pi}{2}}(\cos x)dx$

$=\left[\sin x\right]_0^{\frac{\pi}{2}}$

$=\left[\sin\frac{\pi}{2}-\sin 0\right]=1$

# Average Value of Function

**Part - 18**

## Average Value of Function

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by $< y >= \frac{\int_a^b y dx}{\int_a^b dx} = \frac{\int_a^b y dx}{b-a}$

Suppose there is a function y = f(x)

Then average value of y = f(x) from $x_1$ = a to $x_2$ = b is

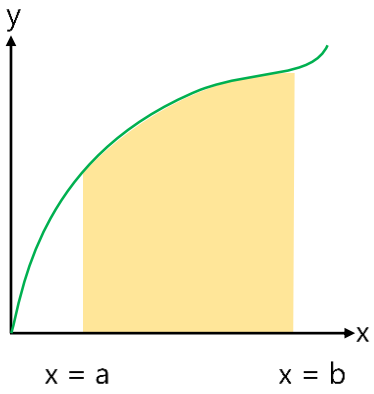
$$< y >= \frac{\int_{x_1}^{x_2} f(x)dx}{\int_{x_1}^{x_2} dx}$$

$$< y >= \frac{\int_a^b f(x)dx}{\int_a^b dx}$$

## Average Value of Function

In case of Graphical section : - The average value of y = f(x) from $x_1$ = a to $x_2$ = b is

$$< y > \frac{\text{area under } y-x \text{ curve}}{\text{range of } x\ (b-a)}$$




### Illustration 1.

What is the average value of the function $x^3$ on the interval [0, 4] ?

**Solution.**

Average value = $\frac{\int_0^4 x^3 dx}{4-0} = \frac{\left[\frac{x^4}{4}\right]_0^4}{4}$

$= \frac{256}{16} = 16$ unit

**Illustration 2.**

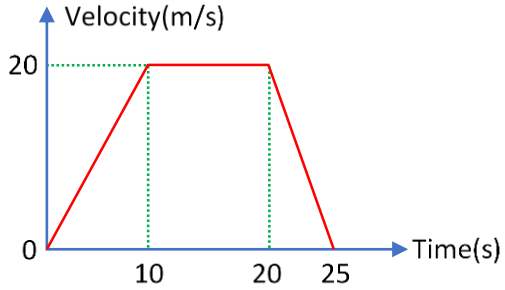
Find average value of a function I = $i_0$cost in interval of time [0, $\frac{\pi}{2}$]

**Solution.**

$$<I>=\frac{\int_0^{\pi/2}(I_0\cos t)dt}{\frac{\pi}{2}-0}=\frac{I_0\int_0^{\pi/2}\cos t dt}{\frac{\pi}{2}}=\frac{I_0}{\frac{\pi}{2}}=\frac{2I_0}{\pi}\text{ unit}$$

**Illustration 3.**

The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car in first 25 seconds is –




**Solution.**

$$\text{Average velocity} = \frac{\int_0^{25} v dt}{25-0}=\frac{\text{Area of v-t graph between t=0 to t = 25 s}}{25}$$

$$=\frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right]=14\text{m/s}$$

**Illustration 4.**

A particle is moving with velocity, v = ($3t^2 + 4t^3 + 4$)m/s. Find <v> for interval 0 to 2 sec.

**Solution.**

$$<v>=\frac{\int_0^2 v dt}{2-0}=\frac{\int_0^2\left(3t^2+4t^3+4\right)}{2-0}=\frac{\left(\frac{3t^3}{3}+\frac{4t^4}{4}+4t\right)_0^2}{2}$$

$$=\frac{8+16+8}{2}=16\text{ unit}$$

**Illustration 5.**

What is the average value of the function f(t) = cosπt in the interval [0, 1]?

**Solution.**

$$\langle f(t)\rangle=\frac{\int_0^1 f(t)dt}{1-0}=\frac{\int_0^2(\cos\pi t)}{1}=\frac{\left[\sin(\pi t)\right]_0^1}{\pi}=\frac{\left[\sin\pi-\sin(0)\right]}{\pi}=0$$

# Quadratic Equation and Binomial Theorem

**Part - 19**

## Quadratic Equation

An algebraic equation of second order (Highest power of the variable is equal to 2) is called a quadratic equation.

$ax^2 + bx + c = 0$ is the general Quadratic Equation.

where $a \neq 0$

## Roots of Quadratic Equation

The general solution of the quadratic equation or it's roots are:

$$x = \frac{-b \pm \sqrt{D}}{2a} \Rightarrow x_1 = \frac{-b+\sqrt{D}}{2a}, \quad x_2 = \frac{-b-\sqrt{D}}{2a}$$

Where $D = b^2 - 4ac$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

## Condition for Real and Imaginary Roots

| For Real Roots | For Imaginary Roots |
|---|---|
| $D \geq 0$ | $D < 0$ |
| $b^2 - 4ac \geq 0$ | $b^2 - 4ac < 0$ |

## Sum and Product of Roots

| Sum of the roots | Product of the roots |
|---|---|
| $x_1 + x_2 = -\frac{b}{a}$ | $x_1 x_2 = \frac{c}{a}$ |

### Illustration 1.

Solve the equation to $x^2 + 3x - 18 = 0$

**Solution.**

$$x^2 + 3x - 18 = 0$$

$$x = \frac{-3 \pm \sqrt{9-4(1)(-18)}}{2(1)} = \frac{-3 \pm \sqrt{81}}{2}$$

$$x_1 = \frac{-3+9}{2} = 3 \; ; \; x_2 = \frac{-3-9}{2} = -6$$

**Illustration 2.**

Solve for x : $pqx^2 - (p^2 + q^2)\,x + pq = 0$

**Solution.**

$pqx^2 - (p^2 + q^2)\,x + pq = 0$

$$x = \frac{-\left(-\left(p^2+q^2\right)\right) \pm \sqrt{\left(-\left(p^2+q^2\right)\right)^2 - 4(pq)(pq)}}{2(pq)} = \frac{\left(p^2+q^2\right) \pm \left(p^2-q^2\right)}{2pq}$$

$$x_1 = \frac{p}{q} \quad ; \quad x_2 = \frac{q}{p}$$

**Illustration 3.**

In quadratic equation $ax^2 + bx + c = 0$, if discriminant is $D = b^2 - 4ac$, then roots of the quadratic equation are : (choose the correct alternative)

(1) Real and distinct, if $D > 0$

(2) Real and equal (i.e., repeated roots), if $D = 0$.

(3) Non-real (i.e. imaginary), if $D < 0$

(4) All of the above are correct

**Solution.**

(4) All of the above are correct

## Binomial Theorem

An algebraic expression containing two terms is called a binomial expression.

For example : $(a + b)$, $(a + b)^3$, $(2x - 3y)^{-2}$ etc.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\times 1}x^2 + \frac{n(n-1)(n-2)}{3\times 2\times 1}x^2 + \ldots\ldots$$

## Binomial Approximation

If x is very small, compared to 1, then terms containing higher powers of x can be neglected so $(1+x)^n \approx 1 + nx$.

If $|x| < < 1$ — Binomial Approximation →

- $(1+x)^n \approx 1 + nx$
- $(1-x)^n \approx 1 - nx$
- $(1+x)^{-n} \approx 1 - nx$
- $(1-x)^{-n} \approx 1 + nx$

**Illustration 4.**

Find the value of

(i) $(1.01)^4$ (ii) $(0.997)^2$ (iii) $\sqrt{0.99}$

**Solution.**

(i) $(1.01)^4 = (1+0.01)^4 \approx 1 + 4\,(0.01) \approx 1.04$

(ii) $(0.997)^2 = (1 - 0.003)^2 \approx 1 - 2\,(0.003) \approx 0.994$

(iii) $\sqrt{0.99} = (1 - 0.01)^{1/2} \approx 1 - \frac{1}{2}(0.01) \approx 1 - 0.005 \approx 0.995$

**Illustration 5.**

Given that $g=\frac{GM}{(R+h)^2}$, find the value of g if h << R

**Solution.**

$$g=\frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}=\frac{GM}{R^2}\left(1+\frac{h}{R}\right)^{-2}$$

If h << R

$$g=\frac{GM}{R^2}\left(1-\frac{2h}{R}\right)$$

# Logarithm and Progressions

**Part - 20**

## Progression

### Arithmetic Progression (AP)

**General form:** a, (a + d), (a + 2d), ....., [a + (n – 1) d]

Here First Term : a

Common Difference : d

$n_{th}$ Term : $\left[a+(n-1)d\right]$

Sum of first n terms: $S_n=\frac{n}{2}\left[1^{st}\text{ term}+n^{th}\text{ term}\right]$

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

**Illustration 1.**

Find the fifth term of given Arithmetic Progression:

5, 7, 9, . . .

**Solution.**

a = 5

d = 7 – 5 = 9 – 7 = 2

Fifth term = (a + 4d) = (5 + 4 (2)) = 13

**Illustration 2.**

Find the sum of first ten terms of given Arithmetic Progression:

2, 4, 6, . . .

**Solution.**

n = 10

d = 4 – 2 = 6 – 4 = 2

$S_n=\frac{10}{2}\left[2(2)+(10-1)(2)\right]$

= 5 [4 + 18] = 110

**Illustration 3.**

Find the sum of given series:

4 + 8 + 12 + . . . . . + 64

**Solution.**

First term = 4

Last term = 64

$n^{th}$ term = a + (n – 1) d

64 = 4 + (n – 1) 4

n = 16

$S_n=\frac{16}{2}\left[4+64\right]=544$

**Illustration 4.**

Find the sum of first 20 natural numbers:

**Solution.**

First 20 natural numbers = 1, 2, 3, ..... 20

a = 1, d = 1

$S_n = \frac{20}{2}\left[2(1)+(19)(1)\right] = 210$

## Geometric Progression (GP)

**General form :** a, ar, $ar^2$, . . . . . $ar^{(n-1)}$

First Term : a

Common Ration : r

$n^{th}$ Term : $ar^{(n-1)}$

Sum of first n terms : $S_n = \frac{a\left(1-r^n\right)}{(1-r)}$

Sum of all the terms of an infinite GP : $S_\infty = \frac{a}{(1-r)}$ ; Only when |r| < 1

**Illustration 6.**

Find the sixth term of 1, 2, 4, . . . . . .

**Solution.**

Sixth term = $ar^5$

a = 1, r = $\frac{2}{1} = \frac{4}{2} = 2$

$\therefore$ Sixth term = $1(2)^5 = 32$

**Illustration 7.**

Find sum of all the terms of an infinite GP : $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.....\infty$

**Solution.**

a = 1, r = $\frac{1/2}{1} = \frac{1/4}{1/2} = \frac{1}{2}$

$S_n = \frac{a}{1-r} = \frac{1}{1-1/2} = 2$

## Formulae to Remember

Sum of first n natural numbers:

$1 + 2 + 3 + 4 + 5 + ..... + n = \frac{n(n+1)}{2}$

Sum of squares of first n natural numbers:

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + .....+n^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of cubes of first n natural numbers :

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + .....+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

## Logarithm

The exponent or power to which a base must be raised to yield a given number.

Expressed mathematically, x is the logarithm of n to the base b

$x = \log_b n$ (if $b^x = n$, exponential form)

If base is 10 it is called **Standard log** $\Rightarrow \log_{10} n$

If base is e then it is called **Natural log** $\Rightarrow \log_e n$

$e \approx 2.71$

## Common Formulae of Logarithm

Product Formula $\Rightarrow \log (mn) = \log m + \log n$

Quotient Formula $\Rightarrow \log\left(\frac{m}{n}\right) = \log m - \log n$

Power Formula $\Rightarrow \log (m^n) = n \log m$

## Standard Values of Logarithm

Base Changing Formula $\Rightarrow \log_e m = 2.303 \log_{10} m$

For any Base $\Rightarrow \log_b 1 = 0$

For Base a $\Rightarrow \log_a a = 1$

## Standard Values to remember

| Base 10 | Base e |
|---|---|
| log 2 = 0.301 | ln 2 = 0.693 |
| log 3 = 0.477 | ln 3 = 1.09 |
| log 5 = 0.699 | ln 5= 1.6 |

### Illustration 8.

Find the value of :

(i) $\ln e^5$ (ii) $\ln e^{2/3}$

**Solution.**

(i) $\ell n e^5 = 5 \ell n e = 5 \log_e^e = 5$

(ii) $\ell n e^{2/3} = \frac{2}{3} \ell n_e^e = \frac{2}{3}$

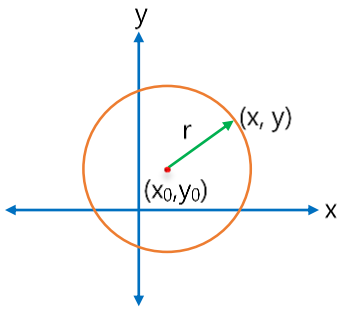
# Graphs - Ellipse, Circle

**Part - 21**

## Circle

Assume that (x, y) are the coordinates of a point on the circle shown, the centre is at $(x_0, y_0)$ and the radius is r.

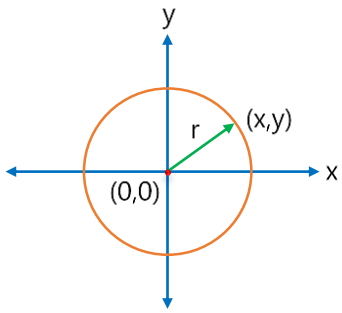
Equation of circle = $(x - x_0)^2 + (y - y_0)^2 = r^2$




## Equation of Circle with center at Origin

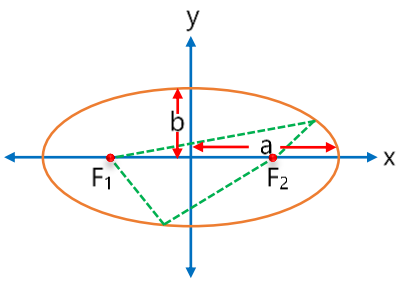
$(x - 0)^2 + (y - 0)^2 = r^2$

$x^2 + y^2 = r^2$




## Ellipse

An ellipse is the locus of points in a plane, the sum of whose distances from two fixed points is a constant value. The two fixed points are called the **foci** of the ellipse.




In this diagram: -

a = semi major axis ; b = semi minor axis

$F_1$ and $F_2$ = foci of the ellipse.

## Equation of Ellipse with center at Origin

The equation of ellipse is written in terms of it's semi-major axis and semi-minor axis as:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

### Illustration 1.

What is the radius of the circle given by:

$x^2 + y^2 = 49$

### Solution.

$x^2 + y^2 = 49$ (given equation) .....(1)

$x^2 + y^2 = r^2$ (general equation) .....(2)

Comparing both equations : $r^2 = 49$

Radius of the circle = r = 7

### Illustration 2.

What is the value of c if the radius of the circle is 9 and centre is at origin:

$x^2 + y^2 = c$

### Solution.

$x^2 + y^2 = c$ (given equation) .....(1)

$x^2 + y^2 = r^2$ (general equation) .....(2)

Comparing both equations : $c = r^2$

Radius of the circle = c = 81

### Illustration 3.

Find length of major axis and minor axis for ellipse

$$\frac{x^2}{16}+\frac{y^2}{36}=1$$

### Solution.

$\frac{x^2}{16}+\frac{y^2}{36}=1$ (given equation) .....(1)

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (general equation) .....(2)

Comparing both equations :

a = 4 (semi minor axis)

b = 6 (semi major axis)

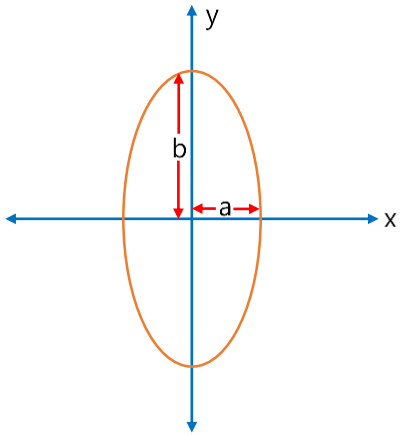


**Illustration 4.**

If the length of major axis is 5 and minor axis is 3 then write the equation of ellipse centered at origin.

**Solution.**

a = 5 ; b = 3

$$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$$

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

# Graphs - Parabola, Rectangular Hyperbola, Exponential Functions

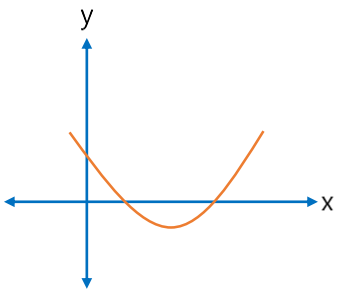
**Part - 22**

## Parabola

The equation of parabola is given by:

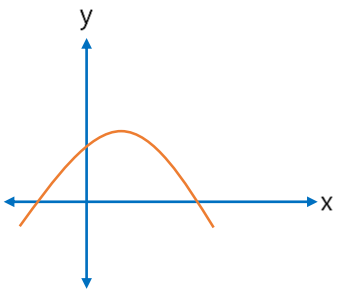
$$y = ax^2 + bx + c$$

If a > 0




**Upward Opening Parabola**

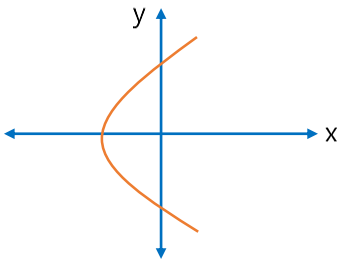
If a < 0




**Downward Opening Parabola**

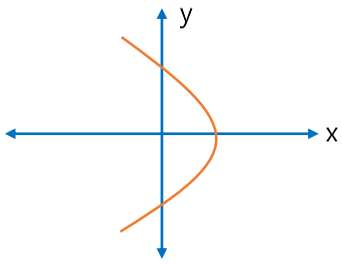
$$x = ay^2 + by + c$$

If a > 0




**Rightward Opening Parabola**

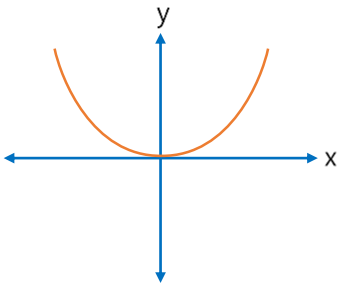
If a < 0



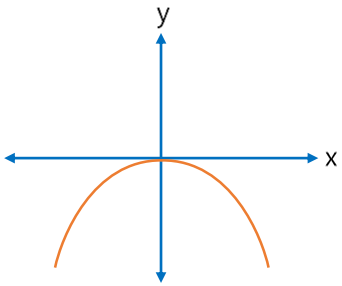

**Leftward Opening Parabola**

## Some standard Parabola

$y = x^2$




$y = -x^2$



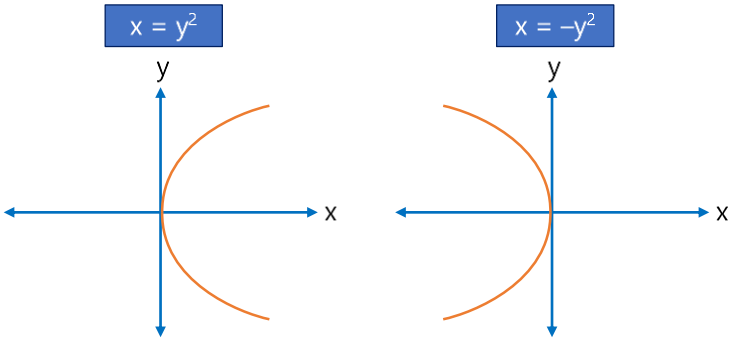



**Illustration 1.**

A particle of mass m is moving with speed v. Draw the graph of K.E vs v

**Solution.**

K.E. = $\frac{1}{2}mv^2$

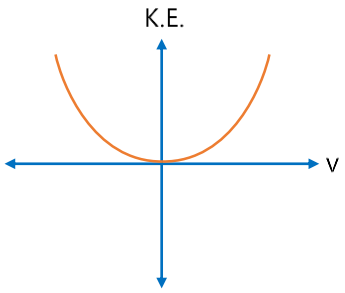



**Illustration 2.**

If $x = 9t^2$ and $y = 3t$ represents the coordinate of a particle, then its path will be?

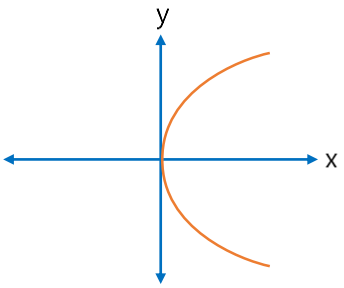
**Solution.**

$x = 9t^2$ .....(1)

$y = 3t$ .....(2)

From equation (2) $t = \frac{y}{3}$ .....(3)

Now, from equation (1) and (3) $x = 9\ \frac{y^2}{9} = y^2$



## Rectangular Hyperbola

The equation of Rectangular Hyperbola is given by:

xy = Constant

$y \propto \frac{1}{x}$

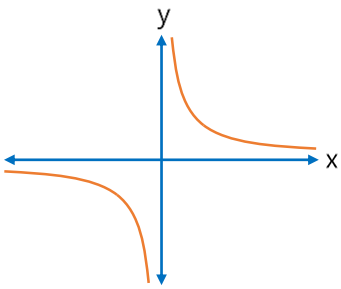


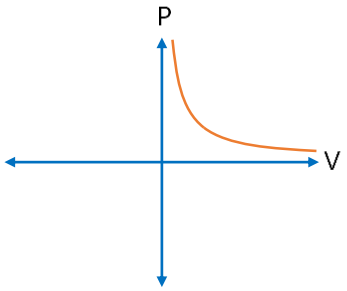

**Illustration 3.**

Draw graph between pressure and volume for an ideal gas at constant temperature (PV = Constant)

**Solution.**

PV = nRT

PV = constant




## Exponential Graphs

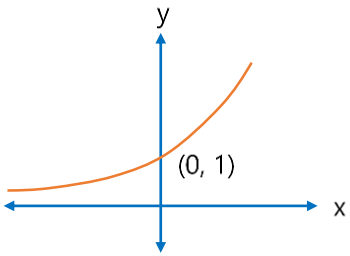
There are two types of exponential graphs:

(i) Exponential Growth

(ii) Exponential Decay

Exponential Growth

$y = e^{kx}$




Exponential Decay

$y = e^{-kx}$

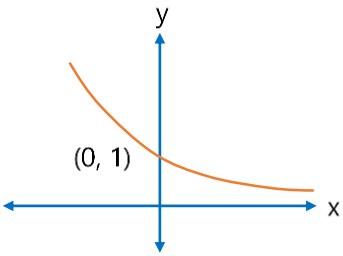


**Illustration 4.**

A particle moves along path $y = 9x^2 - 2x + 4$, then its path will be?

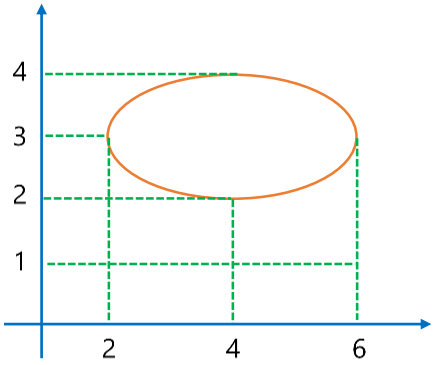
**Solution.**

$y = ax^2 - bx + c$

$\therefore$ path will be parabola

**Illustration 5.**

Calculate the area enclosed by shown ellipse




**Solution.**

Shaded area = Area of ellipse = $\pi ab$

Here $a = 6 - 4 = 2$ and $b = 4 - 3 = 1$

$\Rightarrow$ Area = $\pi \times 2 \times 1 = 2\pi$ units

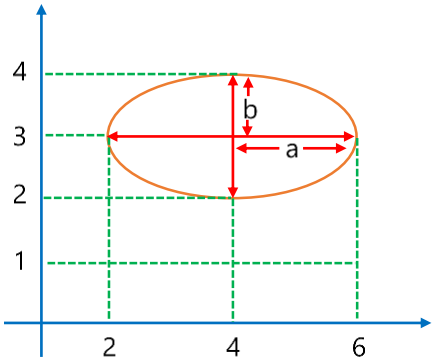
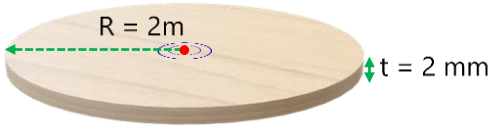



**Illustration 6.**

Calculate the volume of given disc.




**Solution.**

Volume = (Area) (thickness)

Volume = $\pi R^2 t = (3.14)(2)^2\left(2\times10^{-3}\right) = 25.12\times10^{-3}\,m^3$